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A general constitutive framework for the combined creep, plasticity and swelling behavior of nuclear fuels in an implicit hypoelastic formulation

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Abstract

This work is motivated by the need to analyze the behavior of nuclear fuels which under normal operating conditions build up stresses due to non-homogeneous thermal expansion, fission gas and solid product swelling among other phenomena that are simultaneously relaxed by creep and plastic flow. This report details the stress and tangent update equations for combined J_2 based rate independent plasticity, time-hardening creep and fission gas induced swelling effects in a fully implicit hypo-elastic formulation involving two cases: pure creep without plasticity where the yield criterion has not yet been met and the combined effect of both creep and plasticity beyond yield. Closed form expressions for the consistent material tangent to be used in both cases are derived which can be used in implicit codes and is expected to help in obtaining optimal convergence rates.

Keywords: Plasticity, creep, swelling, metallic fuel, material tangent, implicit integration

1. Introduction and motivation

Nuclear fuels exhibit extremely complex behavior during irradiation which include but are not limited to thermal expansion, swelling due to gas and solid products of nuclear fission, phase transformation, creep and plastic flow, damage and cracking. Swelling is a complex process starting from the production of fission gas atoms, migration, coalescence and release as is well documented in Olander [1] (see Table 13.1). Among the hundreds of fission products, gases such as xenon and krypton, due to their low solubility in the fuel, precipitate as bubbles and can greatly affect overall fuel behavior. While fuel swelling is a universal problem encountered in the irradiation of nuclear fuels, there are differences in the way it affects fuel behavior depending on the type of fuel [2]. For example, oxide fuels which have low thermal conductivity operate with high thermal gradients, causing the migration and release of gas bubbles thereby causing little swelling [2]. However ceramic fuels with high thermal conductivity (i.e nitride and carbide fuels) and

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metallic fuels which operate with a relatively low thermal gradient retain fission gas bubbles for longer, causing significant swelling before it is released from the fuel [2]. In addition, metallic fuels often exhibit creep and plastic flow, especially when operating at high temperatures which relax stresses in the fuel built up by other phenomena such as thermal expansion and fission gas swelling.

As there is increasing need to perform full scale 3D analyses over long periods of operation (time scales of weeks to months), there is a need to develop constitutive equations for the material behavior [3]. Moreover to achieve this, implicit finite element codes are being increasingly used so that there is no restriction on the time step as is common in explicit finite element codes. To ensure optimal convergence rates of the global finite element equations, the consistent material tangent should be provided to the finite element solver in accordance with the algorithm used for the stress update. A material tangent evaluated numerically could be used but would also be expensive to compute [4, 5]. A closed form expression for the material tangent would be considerably more computationally efficient than using an explicit finite element solution or evaluating a numerical tangent.

In this work, closed form expressions are derived for the stress and consistent material tangent update for the combined effects of swelling, creep and plasticity for either of two possible cases where the material: (1) is within the yield surface and undergoing creep flow or (2) has yielded and undergoing both creep and plastic flow simultaneously. It is assumed for the present purposes that plastic flow follows an isotropic hardening law, however a different hardening law may be chosen suitably and implemented with a similar approach as outlined in this report.

2. Fission gas swelling model

To simulate swelling in fuels due to fission by-products, a model which governs the radius and consequently other state variables such as temperature and pressure inside the bubble needs to be included. The mechanistic swelling model developed by Matthews et al. which relates these quantities to the stress in the fuel is considered here [6]. As a first approximation of fission gas bubble swelling, all bubbles can be assumed to be the same radius and comprised of xenon gas. The bulk swelling due to fission gas bubbles can be calculated from the bubble radius r_b or alternatively bubble volume V_b and concentration of bubbles per unit volume C_b as

$$V_{sw} = \frac{4}{3}\pi r_b^3 C_b = V_b C_b \tag{1}$$

In order to accurately capture C_b, a nucleation model must be considered that considers local state variables. However, through inspection of fuel cross-sectional micrographs, the concentration of fission gas bubbles seems to be roughly constant within a phase and across different irradiation conditions, with few large bubbles in the -phase, many small bubbles in the β-phase, and many oblong pores in the α-phase. Within this current implementation, $C_b = 10^{13}$ is held fixed at an approximate value estimated for the largest bubble sizes in the γ-phase. Future models would benefit from temperature and phase dependent values of C_b, as well as consideration for the non-spherical porosity in the α-phase. However, the current implementation is intended as a proof-of-concept for a tightly coupled thermo-mechanical-swelling simulation, and thus will make broad assumptions until fully implemented.

The physics behind fission gas bubble swelling consists of modeling fission gas creation in the fuel, diffusion of the individual gas atoms to bubbles, growth of the fission gas bubbles, and bulk volumetric response as porosity increases. The concentration of fission gas atoms in the fuel C_g can be calculated by tracking the introduction of gas atoms through fission and the loss of gas atoms due to absorption to fission gas bubbles. What results is coupled partial differential equations describing the concentration as,

$$\dot{C}_g = \dot{g}_{source} - \dot{g}_{sink} \tag{2}$$

The introduction of fission gas bubbles calculated by multiplying the fission rate density $\dot{F}$ by the yield fraction Y_f.

$$\dot{g}_{source} = \dot{F} Y_f \tag{3}$$

As the fission gas atoms move through the fuel, they will be absorbed by bubbles with concentration C_b and radius r_b, which can be treated as a diffusion limited reaction into a perfect sink [1],

$$\dot{g}_{sink} = 4\pi r_b D_g C_g C_b \tag{4}$$

Here, D_g is the fission gas diffusivity in the fuel, given by,

$$D_g = D_{g0} \exp\left(-\frac{Q_g}{k_B T}\right) \tag{5}$$

where k_B is the Boltzmann constant and T is temperature.

Due to the small diffusivities of the fission gas bubbles, bubble coalescence is assumed to be negligible, with only the diffusivity of the single gas atom through the matrix as the only transport mechanism. The radius of the bubble is a complex function that depends on the surface energy of the material, as well as the local conditions temperature and stress. Following the derivation by Olander [1], the radius of the bubble can be calculated using the van der Waals equation of state to formulate the pressure inside a gas bubble,

$$P_{bubble} = \frac{k_B T}{V_b / A_b - B_{Xe}} \tag{6}$$

where A_b is the number of atoms in the bubble, V_b is the volume of the bubble, and $B = 0.085 nm^3$ is the volume occupied by a single Xe gas atom. The pressure of the bubble is balanced by the surface tension γ_s and the local hydrostatic stress in the material, which can be written as,

$$P_{surface} = \frac{2\gamma_s}{r_b} - p \tag{7}$$

where p is the hydrostatic stress or pressure in the material given by $\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$. Contrary to convention, the 'pressure' in the material is considered to be positive in tension and is used interchangeably with the term hydrostatic stress. Equating (6) and (7), the pressure in the material can be written as,

$$p = \frac{2\gamma_s}{r_b} - \frac{k_B T}{V_b / A_b - B} \tag{8}$$

The evolution of the number of gas atoms in the bubble A_b also depends on the radius of the bubble itself, and is described the equation,

$$\dot{A}_b = 4\pi r_b D_g C_g (1 - \lambda) \tag{9}$$

where D_g and C_g are fission gas diffusivity and the concentration of fission gas atoms in the fuel given by (5) and (2) respectively. And λ is given by,

$$\lambda = \frac{1}{2}\left[1 + \tanh\left(\frac{f - f_{cr}}{f_w}\right)\right] \tag{10}$$

where f_{cr} and f_w are constants which are input to the model and f is the porosity which can be related to the volume and concentration of the bubbles by the relation,

$$f = \frac{V_{sw}}{1 + V_{sw}} = \frac{V_b C_b}{1 + V_b C_b} \tag{11}$$

3. Constitutive response

We start with assuming that all strains are small and the constitutive response is linear elastic, which we describe with a hypoelastic rate formulation. Assume that the total strain tensor $\boldsymbol{\varepsilon}$ can be additively decomposed into an elastic strain $\boldsymbol{\varepsilon}^{el}$, thermal strain $\boldsymbol{\varepsilon}^{th}$, swelling strain $\boldsymbol{\varepsilon}^{sw}$, creep strain $\boldsymbol{\varepsilon}^{cr}$ and plastic strain such that in rate form,

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{el} + \dot{\boldsymbol{\varepsilon}}^{th} + \dot{\boldsymbol{\varepsilon}}^{sw} + \dot{\boldsymbol{\varepsilon}}^{cr} + \dot{\boldsymbol{\varepsilon}}^{pl} \tag{12}$$

The constitutive equation can be expressed in terms of the elastic strain as,

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}^{el} \tag{13}$$

where $\mathbb{C}$ is the fourth order material stiffness tensor which is considered to be constant. Taking the time derivative,

$$\dot{\boldsymbol{\sigma}} = \mathbb{C} : \dot{\boldsymbol{\varepsilon}}^{el} \tag{14}$$

Using (12), write the elastic strain in terms of the total, thermal, swelling, creep and plastic strains,

$$\dot{\boldsymbol{\sigma}} = \mathbb{C} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{th} - \dot{\boldsymbol{\varepsilon}}^{sw} - \dot{\boldsymbol{\varepsilon}}^{cr} - \dot{\boldsymbol{\varepsilon}}^{pl}\right) \tag{15}$$

Using a backward (implicit) Euler integration scheme (see Eqns. (1.4.4), Simo and Hughes [7]), the incremental form of (15) can be written as,

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathbb{C} : \left(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{th} - \Delta\boldsymbol{\varepsilon}^{sw} - \Delta\boldsymbol{\varepsilon}^{cr} - \Delta\boldsymbol{\varepsilon}^{pl}\right)_{n+1} \tag{16}$$

By contraction on both sides of (16) by the volumetric projection tensor $\mathbb{I}^{vol}$ [8], we get the update equations for the (scalar) hydrostatic component of stress (referred to as pressure p here) as,

$$p_{n+1} = p_n + K\left[\mathrm{tr}\left(\Delta\boldsymbol{\varepsilon}\right) - \mathrm{tr}\left(\Delta\boldsymbol{\varepsilon}^{th}\right) - \mathrm{tr}\left(\Delta\boldsymbol{\varepsilon}^{sw}\right)\right]_{n+1} \tag{17}$$

where the tr $(\cdot)$ operator returns the trace of it's argument and K is the bulk modulus. Similarly, contracting with the deviatoric projection tensor $\mathbb{I}^{dev}$, we get the update equations for the deviatoric components of the stress,

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}_n + 2\mu\left(\Delta\boldsymbol{e} - \Delta\boldsymbol{e}^{cr} - \Delta\boldsymbol{e}^{pl}\right)_{n+1} \tag{18}$$

where the deviatoric total strain increment is represented by $\Delta\boldsymbol{e}$ and μ is the shear modulus. The stress tensor can be re-written in terms of pressure and the stress deviator as,

$$\boldsymbol{\sigma}_{n+1} = p_{n+1}\mathbf{1} + \boldsymbol{s}_{n+1} \tag{19}$$

where the tr $(\cdot)$ operator in (17) returns the trace of its argument and $\mathbf{1}$ in (18) is the second order Kronecker delta. The implicit assumption here is that thermal and swelling strains are purely volumetric, and creep and plastic strains are purely deviatoric in nature.

4. Evaluating the hydrostatic stress

Section 2 presents a set of coupled ordinary differential equations relating various quantities involved in the fission gas bubble swelling process such as radius and volume of the bubble, pressure in the material, irradiation related parameters such as fission rate density and transport and the transport and absorption of fission gas atoms in the fuel. The porosity f can be related to the volume and concentration of the bubbles by the relation,

$$f = \frac{V_{sw}}{1 + V_{sw}} = \frac{V_b C_b}{1 + V_b C_b} \tag{20}$$

implying the inverse relation,

$$V_b = \frac{f}{C_b\,(1 - f)} \tag{21}$$

The evolution of porosity with time induces a volumetric swelling strain in the material which is given by the equation [9],

$$\dot{f} = (1 - f)\mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}_{sw}) \tag{22}$$

which in turn can be written in terms of the magnitude of swelling strain as,

$$\dot{\bar{\boldsymbol{\varepsilon}}}_{sw} = \frac{1}{3}\frac{\dot{f}}{1 - f} \tag{23}$$

This can be integrated analytically over a time step from n to $n + 1$ to obtain the incremental change in the swelling strain as,

$$\mathrm{tr}\,(\Delta\boldsymbol{\varepsilon}_{sw}) = \ln\left(\frac{1 - f_n}{1 - f_{n+1}}\right) \tag{24}$$

This can be substituted for the swelling strain in the pressure update (17),

$$p_{n+1} = p_n + K\left[\mathrm{tr}\,(\Delta\boldsymbol{\varepsilon}) - \mathrm{tr}\left(\Delta\boldsymbol{\varepsilon}^{th}\right) - \ln\left(\frac{1 - f_n}{1 - f_{n+1}}\right)\right] \tag{25}$$

At $n + 1$, the pressure inside the bubble from (8) is,

$$p_{n+1} = \frac{2\gamma_s}{r_{b(n+1)}} - \frac{k_B T_{n+1}}{V_{b(n+1)}/A_{b(n+1)} - B_{Xe}} \tag{26}$$

Re-writing (26) as,

$$p_{n+1} r_{b(n+1)}\left(V_{b(n+1)} - B_{Xe}A_{b(n+1)}\right) = 2\gamma_s\left(V_{b(n+1)} - B_{Xe}A_{b(n+1)}\right) - k_B A_{b(n+1)} T_{n+1} r_{b(n+1)} \tag{27}$$

Substituting for the pressure from the constitutive update (25), we have,

$$\begin{aligned}\left[p_n + K_{n+1}\left(\mathrm{tr}\,(\Delta\boldsymbol{\varepsilon}) - \mathrm{tr}\left(\Delta\boldsymbol{\varepsilon}^{th}\right) - \ln\left(\frac{1 - f_n}{1 - f_{n+1}}\right)\right)\right] r_{b(n+1)}\left(V_{b(n+1)} - B_{Xe}A_{b(n+1)}\right) \\ = 2\gamma_s\left(V_{b(n+1)} - B_{Xe}A_{b(n+1)}\right) - k_B A_{b(n+1)} T_{n+1} r_{b(n+1)}\end{aligned} \tag{28}$$

Moving terms in (28) to one side, we write the residual of the equation as,

$$\begin{aligned}\left[p_n + K_{n+1}\left(\mathrm{tr}\,(\Delta\boldsymbol{\varepsilon}) - \mathrm{tr}\left(\Delta\boldsymbol{\varepsilon}^{th}\right) - \ln\left(\frac{1 - f_n}{1 - f_{n+1}}\right)\right)\right] r_{b(n+1)}\left(V_{b(n+1)} - B_{Xe}A_{b(n+1)}\right) \\ -2\gamma_s\left(V_{b(n+1)} - B_{Xe}A_{b(n+1)}\right) + k_B A_{b(n+1)} T_{n+1} r_{b(n+1)} = R\end{aligned} \tag{29}$$

From (10) using the backward Euler scheme, the incremental form of the change in the gas atoms in the bubble A_b can be written as,

$$A_{b(n+1)} = A_{b(n)} + 4\pi r_{b(n+1)} D_{g(n+1)} C_{g(n+1)} \left(1 - \lambda_{n+1}\right) \Delta t \tag{30}$$

where $D_{g(n+1)}$ can be written explicitly in terms of T_{n+1} using (5) and $C_{g(n+1)}$ can again be expressed in incremental form using backward Euler from (2) as,

$$C_{g(n+1)} = \frac{C_{g(n)} + \dot{F} Y_f \Delta t}{1 + 4\pi r_{b(n+1)} D_{g(n+1)} C_b \Delta t} \tag{31}$$

Therefore by knowing the increments of total strain $\Delta\boldsymbol{\varepsilon}$ and thermal strain $\Delta\boldsymbol{\varepsilon}^{th}$, the pressure p_n and parameters B, k_b, f_{cr}, f_w and by relating A_b at $n+1$ to r_b at $n+1$ using (30) and (31), the residual can be written in terms of either the radius, volume or porosity of the bubble at $n+1$. This can be solved numerically to find the stable bubble size at which $R \approx 0$. Since the residual is a polynomial which has multiple roots, using Newton's method is not optimal in this case. We proceed to solve this equation in two steps:

(1) Bracket the smallest value of bubble volume between a lower bound given by the volume of the bubble at time step n and an upper bound determined by marching along the positive axis of bubble volume till there is a change in the sign of the residual given by (29) indicating the presence of a root, and subsequently

(2) Bracket the solution between the bounds and use a bracketing method (such as the bisection method or regula-falsi) to find the root iteratively up to a specified precision.

It is assumed that at the start of the simulation, there are fission gas bubbles created instantaneously with volume equal to that of one xenon atom ($B_{Xe} = 0.085^{-27} m^3$) which serves as the lower bound for bracketing the solution. Consequently, the number of gas atoms in the bubble A_b is taken to be unity, and it is assumed that the C_g is identically zero. It is also assumed for the sake of simplicity that the temperature in the material due to conduction of heat is equal to the temperature of the gas within the bubbles.

This method is used to evaluate the porosity at every time step and consequently update the pressure using (25).

5. Evaluating the deviatoric stress

From (18), the deviatoric stress update equation is written as,

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}_n + 2\mu_{n+1} \left(\Delta\boldsymbol{e} - \Delta\boldsymbol{e}^{cr} - \Delta\boldsymbol{e}^{pl}\right)_{n+1} \tag{32}$$

Assuming that the step from n to $n+1$ is purely an elastic one, and writing the resulting stress as $\boldsymbol{s}^{trial}_{n+1}$.

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - 2\mu_{n+1} \left(\Delta\boldsymbol{e}^{pl}_{n+1} + \Delta\boldsymbol{e}^{cr}_{n+1}\right) \tag{33}$$

where the trial stress is,

$$\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu_{n+1} \Delta\boldsymbol{e}_{n+1} \tag{34}$$

If the plastic and creep strain increments, $\Delta\boldsymbol{e}^{pl}_{n+1}$ and $\Delta\boldsymbol{e}^{cr}_{n+1}$ can be expressed in terms of the actual deviatoric stress $\boldsymbol{s}_{n+1}$, then (33) can be solved for the deviatoric stress iteratively.

5.1. Rate independent plasticity

Under the assumption of classical rate independent associative plasticity, the plastic strain rate is written as

$$\dot{\boldsymbol{e}^{pl}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{35}$$

where $\dot{\lambda}$ is a positive scalar quantity and f is a specified yield function generally written in terms of the deviatoric stress $\boldsymbol{s}$. The incremental form for the plastic strain can be written as,

$$\Delta\boldsymbol{e}^{pl}_{n+1} = \left(\Delta\lambda\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_{n+1} \tag{36}$$

In J_2 plasticity, the yield function is in general is written in terms of the second invariant of the deviatoric stress,

$$J_2 = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} \tag{37}$$

where $\boldsymbol{s}$ is the deviatoric stress defined by by $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{1}$. A common form for the yield function f is,

$$f = \sqrt{2J_2} - \sqrt{\frac{2}{3}}\sigma_Y(\alpha) \tag{38}$$

where $\sqrt{2J_2}$ amounts to the magnitude of the deviatoric stress, $||\boldsymbol{s}|| = \sqrt{\boldsymbol{s} : \boldsymbol{s}}$, σ_Y is the yield stress of the material which can be a function of the accumulated plastic strain α whose evolution is assumed to be,

$$\dot{\alpha} = \dot{\lambda}\sqrt{\frac{2}{3}} \tag{39}$$

the algorithmic counter part of the yield function, written at time $n+1$ is,

$$f_{n+1} = ||\boldsymbol{s}_{n+1}|| - \sqrt{\frac{2}{3}}\sigma_Y(\alpha_{n+1}) \tag{40}$$

where the accumulated plastic strain is given by,

$$\alpha_{n+1} = \alpha_n + \Delta\lambda\sqrt{\frac{2}{3}} \tag{41}$$

Knowing that the second term in the yield function in (40) is a constant yield stress at $n+1$, the gradient of f with respect to the stress is simply,

$$\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_{n+1} = \frac{\partial ||\boldsymbol{s}_{n+1}||}{\partial \boldsymbol{\sigma}_{n+1}} \tag{42}$$

$$= \frac{\boldsymbol{s}_{n+1}}{||\boldsymbol{s}_{n+1}||} \tag{43}$$

$$= \boldsymbol{n}_{n+1} \tag{44}$$

Therefore, substituting (44) in (36), the plastic strain increment can be written as,

$$\Delta\boldsymbol{e}^{pl}_{n+1} = \Delta\lambda\boldsymbol{n}_{n+1} \tag{45}$$

where $\Delta\lambda$ specifies that magnitude of plastic strain increment and $\boldsymbol{n}_{n+1}$ specifies the direction of plastic flow.

5.2. Time-hardening creep

Time dependent creep laws are often written in a strain-hardening or time-hardening form. A law of the latter type is assumed here, where the creep strain rate is,

$$\dot{\boldsymbol{e}}^{cr}_{n+1} = \dot{\bar{e}}^{cr}_{n+1}\,(\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1})\,\tilde{\boldsymbol{n}}_{n+1}(\boldsymbol{s}) \tag{46}$$

The creep strain increment can be written as,

$$\Delta\boldsymbol{e}^{cr}_{n+1} = \Delta\bar{e}^{cr}_{n+1}\,(\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1})\,\tilde{\boldsymbol{n}}_{n+1}(\boldsymbol{s}) \tag{47}$$

where $\Delta\bar{e}^{cr}_{n+1}$ is the magnitude of the creep strain increment which depends on the current von Mises stress, time, temperature and other constants which are material parameters that can be estimated from a uniaxial creep test, and $\tilde{\boldsymbol{n}}_{n+1}(\boldsymbol{s})$ is the direction of flow of creep which is some function of the deviatoric stress. As is commonly done, the direction of creep is assumed to be such that,

$$\tilde{\boldsymbol{n}}_{n+1} = \frac{\partial\tilde{\sigma}_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}} \tag{48}$$

where $\tilde{\sigma}_{n+1}$ is the von Mises stress. To be consistent with the direction of plastic flow, the direction of creep is written in terms of $\boldsymbol{n}$ such that,

$$\Delta\boldsymbol{e}^{cr}_{n+1} = \Delta\bar{e}^{cr}_{n+1}\,(\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1})\,\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1} \tag{49}$$

Consider that a power law for creep strain is assumed in it's 'time hardening' form such that the creep strain rate is given by,

$$\dot{\bar{e}}^{cr} = A\tilde{\sigma}^{m}t^{n}e^{-Q/RT} \tag{50}$$

where $\tilde{\sigma}$ is the von-Mises stress, t is time, Q is the activation energy of the creep mechanism, R is the Boltzmann's constant, T is the absolute temperature and A,m and n are material parameters. Using an implicit time integration scheme, the creep strain increment can be written as,

$$\Delta\bar{e}^{cr}_{n+1} = \Delta t A\tilde{\sigma}^{m}_{n+1}t^{n}_{n+1}e^{-Q/RT_{n+1}} \tag{51}$$

where the material parameter n appearing as the exponential term for time t is not to be confused with the number of the time step appearing in the subscript.

5.3. Deviatoric stress

Substituting (49), (51) and (45) in (33)

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - 2\mu\left[\Delta\lambda\boldsymbol{n}_{n+1} + \left(\Delta t A\tilde{\sigma}^{m}_{n+1}t^{n}_{n+1}e^{-Q/RT_{n+1}}\right)\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1}\right] \tag{52}$$

The unit normal tensor can be evaluated by knowing the trial stress state at $n{+}1$, which can be determined from the state of the material at n and the total applied strain increment ($\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu\,\Delta\boldsymbol{e}_{n+1}$). Both the deviatoric and the trial deviatoric stress, $\boldsymbol{s}_{n+1}$ and $\boldsymbol{s}^{trial}_{n+1}$ can be shown to have the same unit normal tensor $\boldsymbol{n}_{n+1}$ [7], therefore can be determined as,

$$\boldsymbol{n}_{n+1} = \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} = \frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{n+1}\|} \tag{53}$$

Contracting with both sides of (52) with the unit normal tensor $\boldsymbol{n}_{n+1}$, one obtains,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\,\Delta\lambda - 2\mu\,\sqrt{\frac{3}{2}}\left(\Delta t A \tilde{\sigma}^{m}_{n+1} t^{n}_{n+1} e^{-Q/RT_{n+1}}\right) \tag{54}$$

Writing (54) as a residual,

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\,\Delta\lambda + 2\mu\,\sqrt{\frac{3}{2}}\left(\Delta t A \tilde{\sigma}^{m}_{n+1} t^{n}_{n+1} e^{-Q/RT_{n+1}}\right) \tag{55}$$

5.4. *Combined creep and plasticity*

From (55), the residual to evaluate the plastic multiplier in the case of combined creep and plasticity is given by,

$$R = \sqrt{\frac{2}{3}}\sigma_Y - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\,\Delta\lambda + 2\mu\,\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\left(\sigma_Y\right) \tag{56}$$

where the deviatoric stress $\boldsymbol{s}_{n+1}$ must lie on the yield surface i.e.,

$$\|s_{n+1}\| = \sqrt{\frac{2}{3}}\sigma_Y = \sqrt{\frac{2}{3}}\sigma_{Y0} + \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{57}$$

For this case, the residual can be written in terms of the incremental plastic multiplier which is to be determined.

The creep strain increment in terms of the deviatoric stress is,

$$\Delta\bar{e}^{cr}_{n+1} = \Delta t A\left(\sqrt{\frac{3}{2}}\right)^{m}\|s_{n+1}\|^{m}\, t^{n}_{n+1} e^{-Q/RT_{n+1}} \tag{58}$$

which can be written in terms of the yield stress as,

$$\Delta\bar{e}^{cr}_{n+1} = \Delta t A \sigma^{m}_{Y}\, t^{n}_{n+1} e^{-Q/RT_{n+1}} \tag{59}$$

To find the solution to the nonlinear equation given by the residual, the derivative of the creep strain increment with respect to the plastic multiplier is required which can be written as,

$$\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial\Delta\lambda} = \Delta t A m \sigma^{(m-1)}_{Y}\frac{\partial\sigma_Y}{\Delta\lambda} t^{n}_{n+1} e^{-Q/RT_{n+1}} \tag{60}$$

A closed form expression for the derivative of the yield stress can be found if the hardening law is known. For example, considering a linear hardening law, the derivative of the yield stress with respect to the plastic multiplier is,

$$\begin{aligned}\frac{\partial\sigma_Y}{\partial\Delta\lambda} &= \frac{\partial}{\partial\Delta\lambda}\left(\sigma_{Y0} + \tilde{K}\alpha_n + \sqrt{\frac{2}{3}}\tilde{K}\Delta\lambda\right)\\ &= \sqrt{\frac{2}{3}}\tilde{K}\end{aligned} \tag{61}$$

The following terms are then required to find the plastic multiplier and the material tangent,

$$
\begin{aligned}
\Delta\bar{e}^{cr} &= \Delta t A \sigma_Y^m \, t_{n+1}^n e^{-Q/RT_{n+1}} \\
&= \Delta t A \left(\sigma_{Y0} + \tilde{K}\alpha_n + \sqrt{\frac{2}{3}} \tilde{K} \Delta\lambda \right)^m \, t_{n+1}^n e^{-Q/RT_{n+1}} \qquad (62) \\
\frac{\partial \Delta\bar{e}_{n+1}^{cr}}{\partial \Delta\lambda} &= \Delta t A m \sigma_Y^{(m-1)} \frac{\partial \sigma_Y}{\Delta\lambda} t_{n+1}^n e^{(-Q/RT_{n+1})} \\
&= \Delta t A m \sigma_Y^{(m-1)} \sqrt{\frac{2}{3}} \tilde{K} \, t_{n+1}^n e^{-Q/RT_{n+1}} \qquad (63)
\end{aligned}
$$

5.5. *Pure creep*

In the case of creep without any plastic effects, the nonlinear equation to be solved to evaluate the deviatoric stress at the current time step reduces to,

$$
R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}_{n+1}^{trial}\right\| + 2\mu \sqrt{\frac{3}{2}} \left(\Delta t A \tilde{\sigma}_{n+1}^m t_{n+1}^n e^{-Q/RT_{n+1}} \right) \qquad (64)
$$

The von-Mises stress can be written in terms of the deviatoric stress as,

$$
\tilde{\sigma}_{n+1} = \sqrt{\frac{3}{2}} \|s_{n+1}\| \qquad (65)
$$

Substituting into Eqn, (64),

$$
R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}_{n+1}^{trial}\right\| + 2\mu \sqrt{\frac{3}{2}} \left[\Delta t A \left(\sqrt{\frac{3}{2}} \right)^m \|s_{n+1}\|^m t_{n+1}^n e^{-Q/RT_{n+1}} \right] \qquad (66)
$$

The residual is written in terms of a single unknown variable, $\|s_{n+1}\|$ and can be evaluated using a Newton-Raphson iterative scheme. To evaluate the deviatoric stress and the material tangent, the creep strain increment and the derivative of the creep strain increment are required which are respectively given by,

$$
\Delta\bar{e}^{cr} = \Delta t A \left(\sqrt{\frac{3}{2}} \right)^m \|s_{n+1}\|^m t_{n+1}^n e^{-Q/RT_{n+1}} \qquad (67)
$$

$$
\frac{\partial \Delta\bar{e}^{cr}}{\partial \|s_{n+1}\|} = \Delta t A \left(\sqrt{\frac{3}{2}} \right)^m m \|s_{n+1}\|^{(m-1)} t_{n+1}^n e^{-Q/RT_{n+1}} \qquad (68)
$$

6. Consistent Material Tangent

While finite element codes need only a stress update model to reach the correct solution, a Jacobian matrix, also known as the tangent modulus or tangent stiffness greatly affects the rate of convergence. While the tangent modulus has no effect on the accuracy of the solution, it is used to search for the displacement fields that satisfy the weak form of the equilibrium equations [5]. In this case, for the purposes of computational efficiency, since it is important that large time steps be used - on the order of days and weeks in which there can be significant inelastic effects, it is worthwhile to invest in deriving the analytical form of the tangent modulus.

Starting with the constitutive equation (13) at the $n+1^{th}$ time step,

$$\boldsymbol{\sigma}_{n+1} = \mathbb{C} : \boldsymbol{\varepsilon}^{el}_{n+1} \tag{69}$$

we can decompose the R.H.S in terms of the hydrostatic and deviatoric response as,

$$\boldsymbol{\sigma}_{n+1} = K \operatorname{tr}\left(\boldsymbol{\varepsilon}^{el}_{n+1}\right)\mathbf{1} + 2\mu\, \boldsymbol{e}^{el}_{n+1} \tag{70}$$

where $\boldsymbol{e}^{el}_{n+1}$ is the deviatoric component of the elastic strain tensor. Writing (70) in differential form we have,

$$d\boldsymbol{\sigma}_{n+1} = K \operatorname{tr}\left(d\boldsymbol{\varepsilon}^{el}_{n+1}\right)\mathbf{1} + 2\mu\, d\boldsymbol{e}^{el}_{n+1} \tag{71}$$

which can be written in terms of the total, thermal, creep and swelling strains as,

$$d\boldsymbol{\sigma}_{n+1} = K \operatorname{tr}\left(d\boldsymbol{\varepsilon}_{n+1} - d\boldsymbol{\varepsilon}^{th}_{n+1} - d\boldsymbol{\varepsilon}^{sw}_{n+1}\right)\mathbf{1} + 2\mu\, \left(d\boldsymbol{e}_{n+1} - d\boldsymbol{e}^{cr}_{n+1} - d\boldsymbol{e}^{pl}_{n+1}\right) \tag{72}$$

Decomposing the thermal, swelling, creep and plastic strains as,

$$d\boldsymbol{\varepsilon}^{th}_{n+1} = d\boldsymbol{\varepsilon}^{th}_{n} + d\Delta\boldsymbol{\varepsilon}^{th}_{n+1} \tag{73}$$
$$d\boldsymbol{\varepsilon}^{sw}_{n+1} = d\boldsymbol{\varepsilon}^{sw}_{n} + d\Delta\boldsymbol{\varepsilon}^{sw}_{n+1} \tag{74}$$
$$d\boldsymbol{\varepsilon}^{cr}_{n+1} = d\boldsymbol{\varepsilon}^{cr}_{n} + d\Delta\boldsymbol{\varepsilon}^{cr}_{n+1} \tag{75}$$
$$d\boldsymbol{\varepsilon}^{pl}_{n+1} = d\boldsymbol{\varepsilon}^{pl}_{n} + d\Delta\boldsymbol{\varepsilon}^{pl}_{n+1} \tag{76}$$

we can rewrite (126) in terms of an elastic step from n to $n+1$ and an inelastic correction,

$$d\boldsymbol{\sigma}_{n+1} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} - K \operatorname{tr}\left(d\Delta\boldsymbol{\varepsilon}^{th}_{n+1} + d\Delta\boldsymbol{\varepsilon}^{sw}_{n+1}\right)\mathbf{1} - 2\mu\, \left(d\Delta\boldsymbol{\varepsilon}^{cr}_{n+1} - d\Delta\boldsymbol{\varepsilon}^{pl}_{n+1}\right) \tag{77}$$

By re-arranging in terms of the partial derivatives of the total strain, this can be written as,

$$d\boldsymbol{\sigma}_{n+1} = \left(\mathbb{C} - K\frac{\partial}{\partial\boldsymbol{\varepsilon}_{n+1}}\operatorname{tr}\left(\Delta\boldsymbol{\varepsilon}^{th}_{n+1}\right)\mathbf{1} - K\frac{\partial}{\partial\boldsymbol{\varepsilon}_{n+1}}\operatorname{tr}\left(\Delta\boldsymbol{\varepsilon}^{sw}_{n+1}\right)\mathbf{1} - 2\mu\frac{\partial\Delta\boldsymbol{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial\Delta\boldsymbol{\varepsilon}^{pl}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) : d\boldsymbol{\varepsilon}_{n+1} \tag{78}$$

Noting that the thermal strain is only a function of the coefficient of thermal expansion (which is considered to be a constant for now) and the change in temperature, and substituting $\operatorname{tr}(\Delta\boldsymbol{\varepsilon}^{sw}) = \Delta\bar{\boldsymbol{\varepsilon}}^{sw}$,

$$d\boldsymbol{\sigma}_{n+1} = \left(\mathbb{C} - K\frac{\partial(\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}\mathbf{1})}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial\Delta\boldsymbol{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial\Delta\boldsymbol{\varepsilon}^{pl}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) : d\boldsymbol{\varepsilon}_{n+1} \tag{79}$$

Therefore, the constitutive response in terms of the total applied strain can be written as,

$$d\boldsymbol{\sigma}_{n+1} = \mathbb{J} : d\boldsymbol{\varepsilon}_{n+1} \tag{80}$$

where the material Jacobian or the consistent material tangent is given by,

$$\mathbb{J} = \left(\mathbb{C} - K\frac{\partial(\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}\mathbf{1})}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial\Delta\boldsymbol{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial\Delta\boldsymbol{\varepsilon}^{pl}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) \tag{81}$$

or,

$$\mathbb{J} = \left(\mathbb{C} - \mathbb{K}^{sw} - \mathbb{K}^{cr} - \mathbb{K}^{pl}\right) \tag{82}$$

This is the general form of the tangent modulus with creep, plasticity and swelling effects. Note that if there is no swelling, creep or plasticity, the tangent modulus reduces to

$$\mathbb{J} = \mathbb{C} \tag{83}$$

The derivatives $\partial(\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}\mathbf{1})/\partial\boldsymbol{\varepsilon}_{n+1}$, $\partial\Delta\boldsymbol{e}^{cr}_{n+1}/\partial\boldsymbol{\varepsilon}_{n+1}$, $\partial\Delta\boldsymbol{\varepsilon}^{pl}_{n+1}/\partial\boldsymbol{\varepsilon}_{n+1}$ need to be estimated at every time step to evaluate the Jacobian.

6.1. Contribution of swelling

Note that identity tensor **1** is a constant, therefore the first term becomes

$$\frac{\partial\left(\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}\mathbf{1}\right)}{\partial\boldsymbol{\varepsilon}_{n+1}} = \mathbf{1}\otimes\frac{\partial\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{84}$$

We then proceed by first using the chain rule to write

$$\frac{\partial\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{\partial\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+1}}{\partial r_{b(n+1)}}\frac{\partial r_{b(n+1)}}{\partial p_{n+1}}\frac{\partial p_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{85}$$

The first term on the R.H.S can be determined from (24) as

$$\frac{\partial\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}}{\partial f_{n+1}} = \frac{1}{1-f_{n+1}} \tag{86}$$

The second term can be determined from (20) as

$$\frac{\partial f_{n+1}}{\partial r_{b(n+1)}} = \frac{4\pi C_b\left(1-f_{n+1}\right)r^2_{b(n+1)}}{1+V_{b(n+1)}C_b} \tag{87}$$

The third term is a scalar determined from the bubble equilibrium equation given by (27) which relates the bubble radius to the hydrostatic stress or the pressure, and is set equal to a constant α for the present,

$$\frac{\partial r_{b(n+1)}}{\partial p_{n+1}} = \alpha \tag{88}$$

The fourth term can be written as

$$\frac{\partial p_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}} = \frac{\mathbf{1}}{3} \tag{89}$$

Substituting (85) - (89) in (84), we have

$$\frac{\partial(\Delta\bar{\boldsymbol{\varepsilon}}^{sw}_{n+1}\mathbf{1})}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{4\pi C_b r^2_{b(n+1)}}{1+V_{b(n+1)}C_b}\alpha\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{90}$$

where $\frac{1}{3}\mathbf{1}\otimes\mathbf{1}$ is nothing but the volumetric projection tensor [8]. With some manipulation this can also be written as

$$\frac{\partial(\Delta\bar{\varepsilon}^{sw}_{n+1}\mathbf{1})}{\partial\boldsymbol{\varepsilon}_{n+1}} = \alpha\frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}\frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{91}$$

Therefore, the contribution of swelling to the material tangent can be written as

$$\mathbb{K}^{sw}_{n+1} = K\left(\alpha\frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}\frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) \tag{92}$$

Now depending on whether the yield criteria has been satisfied or not, there are two possible cases for the consistent material tangent, case I: combined creep and plasticity and case II: pure creep.

6.2. Case I: Combined creep and plasticity

From (79) and (92), the stress update is given by,

$$d\boldsymbol{\sigma}_{n+1} = \left[\mathbb{C} - K\left(\alpha \frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right) - 2\mu\frac{\partial \Delta \boldsymbol{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial \Delta \boldsymbol{\varepsilon}^{pl}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right] : d\boldsymbol{\varepsilon}_{n+1} \tag{93}$$

From (45), the plastic strain increment can be written as,

$$\Delta \boldsymbol{e}^{pl}_{n+1} = \Delta\lambda \boldsymbol{n}_{n+1} \tag{94}$$

Differentiating,

$$\begin{aligned}\frac{\partial \Delta \boldsymbol{e}^{pl}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} &= \frac{\partial(\Delta\lambda \boldsymbol{n}_{n+1})}{\partial \boldsymbol{\varepsilon}_{n+1}} \\ &= \boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} + \Delta\lambda\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\end{aligned} \tag{95}$$

Therefore, the term $\mathbb{K}^{pl}$ can be written as,

$$\mathbb{K}^{pl} = 2\mu\left(\boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} + \Delta\lambda\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right) \tag{96}$$

Similarly from (49),

$$\frac{\partial \Delta \boldsymbol{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \sqrt{\frac{3}{2}}\left(\boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} + \Delta \bar{e}^{cr}_{n+1}\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right) \tag{97}$$

Similar to (96), the term $\mathbb{K}^{cr}$ can be written as,

$$\mathbb{K}^{cr} = 2\mu\sqrt{\frac{3}{2}}\left(\boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} + \Delta \bar{e}^{cr}_{n+1}\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right) \tag{98}$$

Substituting (95) and (97) in (93) and simplifying,

$$d\boldsymbol{\sigma}_{n+1} = \left[\mathbb{C} - K\left(\alpha \frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right) - 2\mu\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\left(\Delta\lambda + \sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\right) - 2\mu\boldsymbol{n}_{n+1} \otimes \left(\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} + \sqrt{\frac{3}{2}}\frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right)\right] : d\boldsymbol{\varepsilon}_{n+1} \tag{99}$$

in which matrices $\mathbb{K}^{sw}$, $\mathbb{K}^{cr}$ and $\mathbb{K}^{pl}$ that account for the modification of the material stiffness matrix $\mathbb{C}$ due to inelastic effects have to be determined from the trial stress state at time $n+1$. To find these matrices the unit normal tensor $\boldsymbol{n}_{n+1}$, plastic strain increment $\Delta\lambda$, creep strain increment $\Delta\bar{e}^{cr}_{n+1}$ and the respective partial derivatives with respect to the total applied strain $\boldsymbol{\varepsilon}_{n+1}$ need to be evaluated.

6.2.1. Evaluating the unit normal tensor and inelastic strain increments

The unit normal tensor can be evaluated by knowing the trial stress state at $n+1$, which can be determined from the state of the material at n and the total applied strain increment ($\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu(\Delta\boldsymbol{e}_{n+1})$). Both the deviatoric and the trial deviatoric stress, $\boldsymbol{s}_{n+1}$ and $\boldsymbol{s}^{trial}_{n+1}$ can be shown to have the same unit normal tensor $\boldsymbol{n}_{n+1}$ [7], therefore can be determined as,

$$\boldsymbol{n}_{n+1} = \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \tag{100}$$

To determine the inelastic strain increments, one starts with the yield function (40), assuming once again a linear isotropic hardening law, can be written in terms of the yield stress in the unhardened state σ_{Y0} and the hardening term $K(\alpha_{n+1})$ as,

$$f_{n+1} = \|\boldsymbol{s}_{n+1}\| - \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{101}$$

where α_{n+1} is the accumulated hardening strain at time $n+1$. If the material has yielded, the stress state must lie on the yield surface, or in other words, f_{n+1} must be zero which means,

$$0 = \|\boldsymbol{s}_{n+1}\| - \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{102}$$

$$\|\boldsymbol{s}_{n+1}\| = \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{103}$$

From the trial deviatoric stress state in (34),

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - 2\mu\left(\Delta\boldsymbol{e}^{pl}_{n+1} + \Delta\boldsymbol{e}^{cr}_{n+1}\right) \tag{104}$$

Substituting in (104) from Eqs. (94) and (49) and contracting with both sides with the unit normal tensor $\boldsymbol{n}_{n+1}$, one obtains,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\Delta\lambda - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1} \tag{105}$$

It should also be noted that the magnitude of increment in creep strain here depends on a few variables one of which is the stress state at $n+1$, particularly the von-Mises stress, which indirectly makes it a function of the deviatoric stress at $n+1$. Therefore (105) should ideally be written as,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\Delta\lambda - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}(\|\boldsymbol{s}_{n+1}\|) \tag{106}$$

which is to say this is a nonlinear equation in $\|\boldsymbol{s}_{n+1}\|$. Substituting (103) in (105), meaning the deviatoric stress at $n+1$ should lie on the yield surface, one obtains,

$$\sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\Delta\lambda - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\left(\sqrt{\frac{2}{3}}\sigma_{Y0} + \sqrt{\frac{2}{3}}K(\alpha_{n+1})\right) \tag{107}$$

This equation can be solved for the plastic multiplier $\Delta\lambda$ numerically by an iterative Newton-Raphson scheme by writing the residual as,

$$R = \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\Delta\lambda + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\left(\sqrt{\frac{2}{3}}\sigma_{Y0} + \sqrt{\frac{2}{3}}K(\alpha_{n+1})\right) \tag{108}$$

and solving $R \approx 0$, provided an explicit expression is known for the isotropic hardening variable K (e.g. a linear isotropic hardening law can be assumed such that $K(\alpha_{n+1}) = \tilde{K}\alpha_{n+1} = \tilde{K}(\alpha_n + \sqrt{\frac{2}{3}}\Delta\lambda)$) so that the evolution of the yield surface is known in terms of the plastic multiplier. Once the plastic multiplier $\Delta\lambda$ is solved for and the accumulated plastic strain α_{n+1} is known, the creep strain increment $\bar{e}^{cr}_{n+1}$ can be evaluated. Other terms which are needed to evaluate the consistent material tangent are the partial derivatives $\partial\boldsymbol{n}_{n+1}/\partial\boldsymbol{\varepsilon}_{n+1}$, $\partial\Delta\lambda/\partial\boldsymbol{\varepsilon}_{n+1}$ and $\partial\Delta\bar{e}^{cr}_{n+1}/\partial\boldsymbol{\varepsilon}_{n+1}$.

Derivative of the plastic multiplier. The derivative of the plastic multiplier with respect to the total applied strain can be obtained from differentiating the residual in (108),

$$\frac{\partial R}{\partial \boldsymbol{\varepsilon}_{n+1}} = 0 = \sqrt{\frac{2}{3}}\frac{\partial K(\alpha_{n+1})}{\partial \alpha_{n+1}}\frac{\partial \alpha_{n+1}}{\partial \Delta\lambda}\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} - \frac{\partial \left\|\boldsymbol{s}^{trial}_{n+1}\right\|}{\partial \boldsymbol{\varepsilon}_{n+1}} + 2\mu\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} + 2\mu\sqrt{\frac{3}{2}}\frac{\partial \Delta\bar{\varepsilon}^{cr}_{n+1}}{\partial \alpha_{n+1}}\frac{\partial \alpha_{n+1}}{\partial \Delta\lambda}\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{109}$$

$$0 = \sqrt{\frac{2}{3}}K'(\alpha_{n+1})\sqrt{\frac{2}{3}}\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} - 2\mu\boldsymbol{n}_{n+1} + 2\mu\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} + 2\mu\sqrt{\frac{3}{2}}\Delta\tilde{\varepsilon}^{cr'}_{n+1}\sqrt{\frac{2}{3}}\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{110}$$

where the primes quantities $(.)'$ indicated the derivative with respect to the accumulated plastic strain α_{n+1}. Collecting and rearranging terms, one obtains,

$$\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}}\left(\frac{2}{3}K'(\alpha_{n+1}) + 2\mu\Delta\tilde{\varepsilon}^{cr'}_{n+1} + 2\mu\right) = 2\mu\boldsymbol{n}_{n+1} \tag{111}$$

$$\frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\boldsymbol{n}_{n+1}}{\left[1 + \Delta\tilde{\varepsilon}^{cr'}_{n+1} + \frac{K'(\alpha_{n+1})}{3\mu}\right]} \tag{112}$$

Derivative of the unit normal tensor. From (100), the derivative of the unit normal tensor can be written as,

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{1}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|}\frac{\partial \boldsymbol{s}^{trial}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} - \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|^2}\frac{\partial \left\|\boldsymbol{s}^{trial}_{n+1}\right\|}{\partial \alpha_{n+1}}\frac{\partial \alpha_{n+1}}{\partial \Delta\lambda} \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{113}$$

Knowing that the trial deviatoric stress state is $\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu\Delta\boldsymbol{e}_{n+1}$, where $\Delta\boldsymbol{e}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{e}_n$ and $\boldsymbol{e}_{n+1} = \mathbb{I}^{dev}\boldsymbol{\varepsilon}_{n+1}$, the derivative can be written as,

$$\frac{\partial \boldsymbol{s}^{trial}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \boldsymbol{s}^{trial}_{n+1}}{\partial \Delta\boldsymbol{e}_{n+1}}\frac{\partial \Delta\boldsymbol{e}_{n+1}}{\partial \boldsymbol{e}_{n+1}}\frac{\partial \boldsymbol{e}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{114}$$

$$= 2\mu\,\mathbb{I}^{dev} \tag{115}$$

where $\mathbb{I}^{dev}$ is the deviatoric projection tensor [8], $\mathbb{I}^{dev} = \left[\mathbb{I}^{sym} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}\right]$

The expression for the derivative of the magnitude of the trial deviatoric stress with respect to the accumulated plastic strain can be obtained by differentiating the expression for the residual in (108),

$$\frac{\partial \left\|\boldsymbol{s}^{trial}_{n+1}\right\|}{\partial \alpha_{n+1}} = \sqrt{\frac{2}{3}}K'(\alpha_{n+1}) + 2\mu\frac{\partial \Delta\lambda}{\partial \alpha_{n+1}} + 2\mu\sqrt{\frac{3}{2}}\Delta\tilde{\varepsilon}^{cr'}_{n+1} \tag{116}$$

$$= \sqrt{\frac{2}{3}}K'(\alpha_{n+1}) + 2\mu\sqrt{\frac{3}{2}} + 2\mu\sqrt{\frac{3}{2}}\Delta\tilde{\varepsilon}^{cr'}_{n+1} \tag{117}$$

where the term $\frac{\partial \Delta\lambda}{\partial \alpha_{n+1}}$ can be evaluated to be $\sqrt{\frac{3}{2}}$, therefore the term $\frac{\partial \alpha_{n+1}}{\partial \Delta\lambda}$ is $\sqrt{\frac{2}{3}}$. Substituting in (113),

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{2\mu\,\mathbb{I}^{dev}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} - \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|^2}\left[\frac{2}{3}K'(\alpha_{n+1}) + 2\mu\Delta\tilde{\varepsilon}^{cr'}_{n+1} + 2\mu\right] \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{118}$$

$$= \frac{2\mu\,\mathbb{I}^{dev}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} - 2\mu\frac{\boldsymbol{n}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|}\left[1 + \Delta\tilde{\varepsilon}^{cr'}_{n+1} + \frac{K'(\alpha_{n+1})}{3\mu}\right] \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{119}$$

In light of the expression (112), (119) becomes,

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{2\mu\, \mathbb{I}^{dev}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} - \frac{2\mu}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \tag{120}$$

$$= \frac{2\mu}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\mathbb{I}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right) \tag{121}$$

Derivative of the creep strain increment. Finally, the derivative of the creep strain increment with respect to the total applied strain can be written as,

$$\frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \alpha_{n+1}} \frac{\partial \alpha_{n+1}}{\partial \Delta\lambda} \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{122}$$

$$= \Delta\tilde{\varepsilon}^{cr\prime}_{n+1} \sqrt{\frac{2}{3}} \frac{\boldsymbol{n}_{n+1}}{\left[1 + \Delta\tilde{\varepsilon}^{cr\prime}_{n+1} + \frac{K'(\alpha_{n+1})}{3\mu}\right]} \tag{123}$$

Assembling the consistent material tangent. Substituting Eqs. (112), (120) and (123) in (99) and rearranging,

$$\begin{aligned} d\boldsymbol{\sigma}_{n+1} = \Bigg[\mathbb{C} - K\left(\alpha \frac{3 f_{n+1}}{r_{b(n+1)}} \mathbb{I}^{vol} \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right) \\ - (2\mu)^2 \frac{\mathbb{I}^{dev}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1}\right) \\ + (2\mu)^2 \frac{1}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1}\right) \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \\ - \frac{2\mu}{\gamma} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} - \frac{2\mu}{\gamma} \Delta\tilde{\varepsilon}^{cr\prime}_{n+1} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\Bigg] : d\boldsymbol{\varepsilon}_{n+1} \end{aligned} \tag{124}$$

Separating out the swelling contribution,

$$\begin{aligned} d\boldsymbol{\sigma}_{n+1} = \Bigg[\mathbb{C} - (2\mu)^2 \frac{\mathbb{I}^{dev}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1}\right) \\ + (2\mu)^2 \frac{1}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1}\right) \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \\ - \frac{2\mu}{\gamma} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} - \frac{2\mu}{\gamma} \Delta\tilde{\varepsilon}^{cr\prime}_{n+1} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\Bigg] : d\boldsymbol{\varepsilon}_{n+1} \\ - K\left(\alpha \frac{3 f_{n+1}}{r_{b(n+1)}} \mathbb{I}^{vol}\right) : d\boldsymbol{\sigma}_{n+1} \end{aligned} \tag{125}$$

Moving the last term to the left hand side,

$$\begin{aligned} \left[\mathbb{I}^{sym} + K\left(\alpha \frac{3 f_{n+1}}{r_{b(n+1)}} \mathbb{I}^{vol}\right)\right] : d\boldsymbol{\sigma}_{n+1} = \Bigg[\mathbb{C} - (2\mu)^2 \frac{\mathbb{I}^{dev}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1}\right) \\ + (2\mu)^2 \frac{1}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1}\right) \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \\ - \frac{2\mu}{\gamma} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} - \frac{2\mu}{\gamma} \Delta\tilde{\varepsilon}^{cr\prime}_{n+1} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\Bigg] : d\boldsymbol{\varepsilon}_{n+1} \end{aligned} \tag{126}$$

or,

$$d\boldsymbol{\sigma}_{n+1} = \mathbb{J} : d\boldsymbol{\varepsilon}_{n+1} \tag{127}$$

such that the material tangent is given by,

$$\mathbb{J} = \mathbb{A}^{-1}\mathbb{B} \tag{128}$$

where,

$$\mathbb{A} = \mathbb{I}^{sym} + K\left(\alpha \frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}\right) \tag{129}$$

$$\mathbb{B} = \mathbb{C} - \frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\mathbb{I}^{dev}\beta + \left(\frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\beta - \frac{2\mu}{\gamma}\Delta\tilde{\varepsilon}_{n+1}^{cr\prime} - \frac{2\mu}{\gamma}\right)\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \tag{130}$$

and $\gamma = \left[1 + \Delta\tilde{\varepsilon}_{n+1}^{cr\prime} + \frac{K'(\alpha_{n+1})}{3\mu}\right]$ and $\beta = \left(\Delta\lambda + \sqrt{\frac{3}{2}}\Delta\bar{e}_{n+1}^{cr}\right)$.

As a sanity check, it can be verified that in the absence of swelling or creep effects, the material tangent reduces to,

$$\mathbb{J}^{ep} = \left[\mathbb{C} - \frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\mathbb{I}^{dev}\Delta\lambda + \left(\frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\Delta\lambda - \frac{2\mu}{\left[1 + \frac{K'(\alpha_{n+1})}{3\mu}\right]}\right)\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right] \tag{131}$$

which is the same as the consistent elastoplastic tangent given in Simo and Hughes [7], pg. 124.

6.3. *Case II: Pure creep*

In the case of pure creep the stress update is,

$$d\boldsymbol{\sigma}_{n+1} = \left[\mathbb{C} - K\left(\alpha \frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}\frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) - 2\mu\frac{\partial\Delta\boldsymbol{e}_{n+1}^{cr}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right] : d\boldsymbol{\varepsilon}_{n+1} \tag{132}$$

The creep strain at $n+1$ depends on the stress state at $n+1$ which is yet to be determined. A time hardening form of the creep strain rate is assumed,

$$\dot{\boldsymbol{\varepsilon}}_{n+1}^{cr} = \dot{\bar{\varepsilon}}_{n+1}^{cr}\left(\bar{\sigma}_{n+1}, T_{n+1}, t_{n+1}\right)\bar{\boldsymbol{n}}_{n+1}(\boldsymbol{s}) \tag{133}$$

such that the creep strain increment then becomes,

$$\Delta\boldsymbol{e}_{n+1}^{cr} = \Delta\bar{e}_{n+1}^{cr}\left(\bar{\sigma}_{n+1}, T_{n+1}, t_{n+1}\right)\bar{\boldsymbol{n}}_{n+1}(\boldsymbol{s}) \tag{134}$$

where $\Delta\bar{e}_{n+1}^{cr}$ is the magnitude of the creep strain increment which depends on the current von Mises stress, time, temperature and other constants which are material parameters that can be estimated from a uniaxial creep test, and $\bar{\boldsymbol{n}}_{n+1}(\boldsymbol{s})$ is the direction of flow of creep which is some function of the deviatoric stress. As is commonly done, the direction of creep is assumed to be such that

$$\bar{\boldsymbol{n}}_{n+1} = \frac{\partial\bar{\sigma}_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}} \tag{135}$$

where $\bar{\sigma}_{n+1}$ is the von Mises stress. However, since the direction of plastic flow is often written in terms of the deviatoric stress, the direction of creep is also written in terms of $\boldsymbol{n}$ for the sake of consistency such that,

$$\Delta\boldsymbol{e}_{n+1}^{cr} = \Delta\bar{e}_{n+1}^{cr}\left(\bar{\sigma}_{n+1}, T_{n+1}, t_{n+1}\right)\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1} \tag{136}$$

Note that the direction of flow written in terms of the von Mises and the deviatoric stress is different only by a multiplication factor of $\sqrt{3/2}$. The derivative of the creep strain increment $\partial \Delta \boldsymbol{e}^{cr}_{n+1} / \partial \boldsymbol{\varepsilon}_{n+1}$ can then be expanded to be written as

$$\frac{\partial \Delta \boldsymbol{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \sqrt{\frac{3}{2}} \Delta \bar{e}^{cr}_{n+1} \frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} + \sqrt{\frac{3}{2}} \boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{137}$$

The partial derivative of the unit normal tensor with respect to the total strain can be written as

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{s}_{n+1}} \frac{\partial \boldsymbol{s}_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}} \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{138}$$

where from Eqn. 3.3.9 in Simo and Hughes [7], (see also Appendix C)

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{s}_{n+1}} = \frac{1}{||\boldsymbol{s}_{n+1}||} \left(\mathbb{I}^{sym} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \right) \tag{139}$$

and (see Jirasek [8])

$$\frac{\partial \boldsymbol{s}_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}} = \mathbb{I}^{dev} \tag{140}$$

where, $\mathbb{I}^{dev}$ is the deviatoric projection tensor given by,

$$\mathbb{I}^{dev} = \left(\mathbb{I}^{sym} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) \tag{141}$$

such that,

$$\boldsymbol{s}_{n+1} = \mathbb{I}^{dev} : \boldsymbol{\sigma}_{n+1} \tag{142}$$

Substituting in (138) and simplifying

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{1}{||\boldsymbol{s}_{n+1}||} \left(\mathbb{I}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \right) \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{143}$$

Similarly,

$$\frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||} \frac{\partial ||\boldsymbol{s}_{n+1}||}{\partial \boldsymbol{\sigma}_{n+1}} \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{144}$$

where $\frac{\partial ||\boldsymbol{s}_{n+1}||}{\partial \boldsymbol{\sigma}_{n+1}} = \boldsymbol{n}_{n+1}$ (see Appendix B) and $\frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||}$ is yet to be determined. Substituting in (144).

$$\frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||} \boldsymbol{n}_{n+1} \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{145}$$

Substituting for $\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}$ and $\frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}$ in (137) and rearranging,

$$\begin{aligned} \frac{\partial \Delta \boldsymbol{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} &= \sqrt{\frac{3}{2}} \frac{\Delta \bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} \left(\mathbb{I}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \right) \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} + \sqrt{\frac{3}{2}} \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||} (\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}) \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \\ &= \sqrt{\frac{3}{2}} \left[\frac{\Delta \bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} \mathbb{I}^{dev} + (\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}) \left(-\frac{\Delta \bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} + \frac{\partial \Delta \bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||} \right) \right] \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \end{aligned} \tag{146}$$

Combining creep and swelling. Substituting (146) in (132)

$$d\boldsymbol{\sigma}_{n+1} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} - K\left(\alpha \frac{3f_{n+1}}{r_{b(n+1)}} \mathbb{I}^{vol}\right) : d\boldsymbol{\sigma}_{n+1}$$
$$-2\mu\sqrt{\frac{3}{2}}\left[\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||}\mathbb{I}^{dev} + (\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1})\left(-\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} + \frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}\right)\right] : d\boldsymbol{\sigma}_{n+1} \tag{147}$$

Moving the second and third term on the R.H.S to the L.H.S and left pre-multiplying on both sides with its inverse, we have,

$$d\boldsymbol{\sigma}_{n+1} = \Bigg[\mathbb{I}^{sym} + K\alpha\frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol}$$
$$+2\mu\sqrt{\frac{3}{2}}\left[\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||}\mathbb{I}^{dev} + (\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1})\left(-\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} + \frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}\right)\right]\Bigg]^{-1}\mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} \tag{148}$$

or,

$$d\boldsymbol{\sigma}_{n+1} = \mathbb{J} : d\boldsymbol{\varepsilon}_{n+1} \tag{149}$$

where the fourth order Jacobian is

$$\mathbb{J} = \Bigg[\mathbb{I}^{sym} + K\alpha\frac{3f_{n+1}}{r_{b(n+1)}}\mathbb{I}^{vol} + 2\mu\sqrt{\frac{3}{2}}\left[\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||}\mathbb{I}^{dev} + (\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1})\left(-\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} + \frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}\right)\right]\Bigg]^{-1}\mathbb{C}_{n+1} \tag{150}$$

....

In both cases - combined creep and plasticity and pure creep, the Jacobian is evaluated at every time step and is a function of the α, the derivative of the bubble radius with respect to the pressure at $n+1$ which can be evaluated to be (see appendix (A))

$$\alpha = \frac{-c_1}{c_2 + c_3 c_4} \tag{151}$$

where the constants are given by

$$c_1 = r_b\left(V_b - BA_b\right) \tag{152}$$
$$c_2 = 4pV_b r_b - 8\pi r_b^2\gamma_s - pBA_b + A_b k_B T \tag{153}$$
$$c_3 = (1-\lambda)\left[4\pi D_g C_g \Delta t - \frac{\left(4\pi r_b D_g C_g \Delta t\right)\left(4\pi D_g C_b \Delta t\right)}{\left(1 + 4\pi r_b D_g C_b \Delta t\right)}\right] \tag{154}$$
$$-\left(4\pi r_b D_g C_g \Delta t\right)\frac{1}{2f_w}\text{sech}^2\left(\frac{f - f_{cr}}{f_w}\right)\frac{3f(1-f)}{r_b} \tag{155}$$
$$c_4 = 2\gamma_s B + k_B r_b T - pBr_b \tag{156}$$

where quantities $V_b, r_b, A_b, C_g, D_g, p$ are evaluated at $n+1$

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Appendix A Derivative of the bubble radius with respect to pressure

From (27), we first move all terms to the L.H.S and for the sake of brevity omit the subscripts $n+1$. All terms that are evolving are evaluated at $n+1$ unless otherwise mentioned and Δ signifies a step from n to $n+1$

$$pr_bV_b - 2\gamma_sV_b - pr_bBA_b + 2\gamma_sBA_b + k_BA_br_bT = 0 \tag{157}$$

Differentiating with respect to p, we have

$$\begin{aligned} &p\frac{16}{3}\pi r_b^3\frac{\partial r_b}{\partial p} + \frac{4}{3}\pi r_b^4 - 8\gamma_s\pi r_b^2\frac{\partial r_b}{\partial p} - A_bBr_b - pA_bB\frac{\partial r_b}{\partial p} \\ &-pBr_b\frac{\partial A_b}{\partial p} + 2\gamma_sB\frac{\partial A_b}{\partial p} + A_bk_BT\frac{\partial r_b}{\partial p} + k_Br_bT\frac{\partial A_b}{\partial p} = 0 \end{aligned} \tag{158}$$

Grouping derivatives of r_b and A_b, we have

$$\frac{\partial r_b}{\partial p}c_2 + \frac{\partial A_b}{\partial p}c_4 + c_1 = 0 \tag{159}$$

where,

$$\begin{aligned} c_2 &= \left(p\frac{16}{3}\pi r_b^3 - 8\gamma_s\pi r_b^2 - pA_bB + A_bk_BT\right) \\ c_4 &= (-pBr_b + 2\gamma_sB + k_Br_bT) \\ c_1 &= \frac{4}{3}\pi r_b^4 - A_bBr_b = 0 \end{aligned} \tag{160}$$

The derivative of A_b, $\partial A_b/\partial p$ needs to be evaluated which is itself a function of the radius r_b, porosity f and gas atoms concentration C_g. Taking the derivative in (30)

$$\frac{\partial A_b}{\partial p} = \left(4\pi D_gC_g\Delta t\frac{\partial r_b}{\partial p} + 4\pi r_bD_g\Delta t\frac{\partial C_g}{\partial p}\right)(1-\lambda) - 4\pi r_bD_gC_g\Delta t\frac{\partial \lambda}{\partial p} \tag{161}$$

where the derivative of λ can be simplified using the chain rule as,

$$\frac{\partial \lambda}{\partial p} = \frac{\partial \lambda}{\partial f}\frac{\partial f}{\partial r_b}\frac{\partial r_b}{\partial p} \tag{162}$$

Using (10) and (87),

$$\frac{\partial \lambda}{\partial p} = \frac{1}{2f_w}\text{sech}^2\left(\frac{f - f_{cr}}{f_w}\right)\frac{3f(1-f)}{r_b}\frac{\partial r_b}{\partial p} \tag{163}$$

$$\tag{164}$$

To evaluate the derivative of C_g, from (31),

$$\frac{\partial C_g}{\partial p} = -\frac{\left(C_{g(n)} + \dot{F}Y_f\Delta t\right)\left(4\pi D_g C_b \Delta t\right)}{\left(1 + 4\pi r_b D_g C_b \Delta t\right)^2}\frac{\partial r_b}{\partial p} \tag{165}$$

where using (31),

$$\frac{\partial C_g}{\partial p} = -\frac{C_g\left(4\pi D_g C_b \Delta t\right)}{\left(1 + 4\pi r_b D_g C_b \Delta t\right)}\frac{\partial r_b}{\partial p} \tag{166}$$

Substituting (163) and (166) in (161), we have

$$\begin{aligned}\frac{\partial A_b}{\partial p} =& (1-\lambda)\, 4\pi D_g C_g \Delta t \frac{\partial r_b}{\partial p} \\ &- (1-\lambda)\frac{\left(4\pi r_b D_g C_g \Delta t\right)\left(4\pi D_g C_b \Delta t\right)}{\left(1 + 4\pi r_b D_g C_b \Delta t\right)}\frac{\partial r_b}{\partial p} \\ &- \left(4\pi r_b D_g C_g \Delta t\right)\frac{1}{2f_w}\text{sech}^2\left(\frac{f - f_{cr}}{f_w}\right)\frac{3f(1-f)}{r_b}\frac{\partial r_b}{\partial p}\end{aligned} \tag{167}$$

Collecting terms,

$$\frac{\partial A_b}{\partial p} = c_3 \frac{\partial r_b}{\partial p} \tag{168}$$

where,

$$\begin{aligned}c_3 = & (1-\lambda)\left[4\pi D_g C_g \Delta t - \frac{\left(4\pi r_b D_g C_g \Delta t\right)\left(4\pi D_g C_b \Delta t\right)}{\left(1 + 4\pi r_b D_g C_b \Delta t\right)}\right] \\ &- \left(4\pi r_b D_g C_g \Delta t\right)\frac{1}{2f_w}\text{sech}^2\left(\frac{f - f_{cr}}{f_w}\right)\frac{3f(1-f)}{r_b}\end{aligned} \tag{169}$$

Substituting (168) in (159),

$$\frac{\partial r_b}{\partial p}c_2 + c_3\frac{\partial r_b}{\partial p}c_4 + c_1 = 0 \tag{170}$$

from which an expression for $\partial r_b / \partial p$ can be generated given by

$$\frac{\partial r_b}{\partial p} = \frac{-c_1}{c_2 + c_3 c_4} \tag{171}$$

Appendix B Unit normal tensor

The derivative of the magnitude of the deviatoric stress is given by

$$\frac{\partial||s||}{\partial\boldsymbol{\sigma}} = \frac{\partial(\sqrt{2J_2})}{\partial\boldsymbol{\sigma}} \tag{172}$$

$$= \frac{1}{2\sqrt{2J_2}}\frac{\partial(2J_2)}{\partial\boldsymbol{\sigma}} \tag{173}$$

$$= \frac{1}{\sqrt{\boldsymbol{s}:\boldsymbol{s}}}\frac{\partial J_2}{\partial\boldsymbol{\sigma}} \tag{174}$$

where J_2 is the second invariant of the deviatoric stress, and can be written in terms of the invariants of the stress tensor as,

$$J_2 = \frac{1}{3}I_1^2 - I_2 \tag{175}$$

The gradient of J_2 can be written as

$$\begin{aligned}\frac{\partial J_2}{\partial\boldsymbol{\sigma}} &= \frac{1}{3}2I_1\frac{\partial I_1}{\partial\boldsymbol{\sigma}} - \frac{\partial I_2}{\partial\boldsymbol{\sigma}}\\ &= \frac{2}{3}I_1\mathbf{1} - (I_1\mathbf{1} - \boldsymbol{\sigma}^{\boldsymbol{T}})\\ &= \boldsymbol{\sigma} - \left(I_1 - \frac{2}{3}I_1\right)\mathbf{1}\\ &= \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})\mathbf{1}\\ &= \boldsymbol{s}\end{aligned} \tag{176}$$

Substituting (176) in (174), the gradient of $||s||$ with respect to $\boldsymbol{\sigma}$ is obtained as,

$$\frac{\partial||s||}{\partial\boldsymbol{\sigma}} = \frac{\boldsymbol{s}}{\sqrt{\boldsymbol{s}:\boldsymbol{s}}} \tag{177}$$

$$= \boldsymbol{n} \tag{178}$$

Appendix C Derivative of the unit normal tensor with the deviatoric stress

The unit normal tensor is given by

$$\boldsymbol{n} = \frac{\boldsymbol{s}}{\sqrt{\boldsymbol{s}:\boldsymbol{s}}} \tag{179}$$

Taking the derivative with respect to the deviatoric stress tensor and changing to index notation

$$\frac{\partial n_{ij}}{\partial s_{kl}} = \frac{\sqrt{\boldsymbol{s}:\boldsymbol{s}}\frac{\partial s_{ij}}{\partial s_{kl}} - s_{ij}\frac{\partial\sqrt{\boldsymbol{s}:\boldsymbol{s}}}{\partial s_{kl}}}{(\sqrt{\boldsymbol{s}:\boldsymbol{s}})^2} \tag{180}$$

where the first derivative in the numerator is the fourth order identity tensor,

$$\frac{\partial s_{ij}}{\partial s_{kl}} = I^{sym}_{ijkl} \tag{181}$$

To evaluate the second derivative in the numerator, first write

$$\sqrt{\boldsymbol{s}:\boldsymbol{s}} = \sqrt{2J_2} \tag{182}$$

Which implies,

$$\begin{aligned}
\frac{\partial \sqrt{\boldsymbol{s}:\boldsymbol{s}}}{\partial s_{kl}} &= \frac{\partial \sqrt{2J_2}}{\partial s_{kl}} \\
&= \frac{\partial \sqrt{2J_2}}{\partial \sigma_{mn}} \frac{\partial \sigma_{mn}}{\partial s_{kl}} \\
&= \frac{\sqrt{2}}{2\sqrt{J_2}} \frac{J_2}{\partial \sigma_{mn}} I^{dev}_{mnkl} \\
&= \frac{1}{\sqrt{2J_2}} s_{mn} I^{dev}_{mnkl} \\
&= \frac{1}{||s||} s_{kl} \\
&= n_{kl}
\end{aligned} \tag{183}$$

Substituting in (180)

$$\begin{aligned}
\frac{\partial n_{ij}}{\partial s_{kl}} &= \frac{\sqrt{\boldsymbol{s}:\boldsymbol{s}} I^{sym}_{ijkl} - s_{ij} n_{kl}}{(\sqrt{\boldsymbol{s}:\boldsymbol{s}})^2} \\
&= \frac{||s|| I^{sym}_{ijkl} - s_{ij} n_{kl}}{||s||^2} \\
&= \frac{I^{sym}_{ijkl} - n_{ij} n_{kl}}{||s||} \\
&= \frac{1}{||s||} \left(I^{sym}_{ijkl} - n_{ij} n_{kl} \right)
\end{aligned} \tag{184}$$

Or in tensor notation, this can be written as

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{s}} = \frac{1}{||s||} (\mathbb{I}^{sym} - \boldsymbol{n} \otimes \boldsymbol{n}) \tag{185}$$

Appendix D Power-Law Creep

Consider that a power law for creep strain is assumed in it's 'time hardening' form such that the creep strain rate is given by,

$$\dot{\varepsilon}^{cr} = A \bar{\sigma}^m t^n e^{-Q/RT} \tag{186}$$

where $\bar{\sigma}$ is the von-Mises stress, t is time, Q is the activation energy of the creep mechanism, R is the Boltzmann's constant, T is the absolute temperature and A,m and n are material parameters. Using an implicit time integration scheme, the creep strain increment can be written as

$$\Delta\bar{\varepsilon}^{cr} = \Delta t A \bar{\sigma}^{m}_{n+1} t^{n}_{n+1} e^{-Q/RT_{n+1}} \tag{187}$$

where the material parameter n appearing as the exponential term for time t is not to be confused with the number of the time step appearing in the subscript.
Using (49), the deviatoric stress update can be written as

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}_n + 2\mu\boldsymbol{e}_{n+1} - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\boldsymbol{n}_{n+1} \tag{188}$$

By contracting (188) with the unit normal tensor $\boldsymbol{n}_{n+1}$, and noting that the direction of the trial deviatoric stress and the deviatoric stress are the same [7], the magnitude of the deviatoric stress can be written in terms of a trial stress and a creep stress increment,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1} \tag{189}$$

where the creep strain increment $\Delta\bar{e}^{cr}_{n+1}$ is some function of the current stress state $\boldsymbol{\sigma}_{n+1}$ and furthermore is generally a function of the current von-Mises stress state or the deviatoric stress state,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}(\|\boldsymbol{s}_{n+1}\|) \tag{190}$$

Therefore, this is in general, a nonlinear equation in $\|\boldsymbol{s}_{n+1}\|$ which can be solved by a Newton-Raphson iterative scheme. If the residual is written as

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}(\|\boldsymbol{s}_{n+1}\|) \tag{191}$$

solving for $R \approx 0$, the quantities $\|\boldsymbol{s}_{n+1}\|$, $\Delta\bar{e}^{cr}_{n+1}$ and $\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial\|\boldsymbol{s}_{n+1}\|}$ can be determined provided an explicit expression for the creep strain increment in terms of the magnitude of deviatoric stress is known. Substituting these quantities in (146), the combined tangent modulus can be obtained from (150). Closed form expressions for the derivative of the creep strain increment assuming a power law creep can be found in D.

In the case of power-law creep, substituting (51) in (191), one obtains

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\sqrt{\frac{3}{2}}\Delta t A\bar{\sigma}^{m}_{n+1}t^{n}_{n+1}e^{-Q/RT_{n+1}} \tag{192}$$

The von-Mises stress can be written in terms of the deviatoric stress as

$$\bar{\sigma}_{n+1} = \sqrt{\frac{3}{2}}\|s_{n+1}\| \tag{193}$$

Substituting into (192),

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\sqrt{\frac{3}{2}}\Delta t A\sqrt{\frac{3}{2}}^{m}\|s_{n+1}\|^{m}t^{n}_{n+1}e^{-Q/RT_{n+1}} \tag{194}$$

To evaluate the deviatoric stress and the tangent modulus, the creep strain increment and the derivative of the creep strain increment are required which are respectively given by

$$\Delta\bar{\varepsilon}^{cr} = \Delta t A \sqrt{\frac{3}{2}}^{m} ||s_{n+1}||^{m} t_{n+1}^{n} e^{-Q/RT_{n+1}} \tag{195}$$

$$\frac{\partial \Delta\bar{\varepsilon}^{cr}}{\partial ||s_{n+1}||} = \Delta t A \sqrt{\frac{3}{2}}^{m} m||s_{n+1}||^{(m-1)} t_{n+1}^{n} e^{-Q/RT_{n+1}} \tag{196}$$